The Pizzeria At The Base Of A Volcano

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Imagine that we are engaged to estimate the enterprise value of a pizzeria located at the base of a volcano. The pizzeria has historically generated very good cash flow but this cash stream has been put in jeopardy by a recent geological report that found that the once dormant volcano is now becoming active. Whereas the geologists cannot pinpoint the exact future date of eruption the report did state that the volcano could erupt at any time with an expected eruption date ten years hence. The surrounding terrain is such that an eruption would not only destroy the pizzeria but also the town in which it resides. If and when the eruption occurs the business will cease to exist. No insurer will insure this risk and it cannot be diversified away. What is the value of this business?

In the scenario above the relevant negative event is the eruption of the volcano and the effect of this event is the total destruction of the business. Given the facts at hand how do we adjust expected future cash flow for both the negative effect of this event and the probability of its occurrence? The Gordon Growth Model (GGM) is a model for determining the intrinsic value of a business based on a future series of dividends (i.e. free cash flow) that grow at a constant rate in perpetuity. We will make a simple adjustment to the GGM such that the model when applied to our pizzeria properly reflects the ramifications of an eruption on enterprise value. **The goal of this exercise will be to provide tools such that the appraiser can extrapolate the volcano-related adjustments applicable to our pizzeria to the valuation of other more common negative events such as the loss of a major customer or the loss of a key person (for reasons other than death)**.

It should be noted that to properly price for event risk both the cash flow effect and the discount rate effect should be considered. In this exercise we will adjust expected cash flows for the effect of the event but will not adjust the discount rate. Technically the discount rate should also be adjusted upwards to account for increased risk (the volcano could erupt tomorrow).

The Base Case Valuation

Before the most recent geological report was issued the general opinion was that the probability of an eruption in the foreseeable future was zero. We will define the base case valuation to be the enterprise value of the pizzeria given no event risk. With no possibility of an eruption the valuation assumptions are (1) year zero annualized cash flow of \$600,000, (2) an annualized cash flow growth rate of 4% and (3) an annualized risk-adjusted discount rate of 20%. Our base case valuation using the Gordon Growth Model is...

$$\text{Base Case Enterprise Value} = \frac{\$600,000 \times 1.04}{0.20 - 0.04} = \$3,900,000 \tag{1}$$

Exponential Arrival Times

The exponential distribution was first used in the study of arrival times. An arrival time is the length of time that we have to wait before the realization of an event. For the pizzeria in our scenario the relevant arrival time will be the length of the time interval that begins on the valuation date and ends when the volcano erupts. The geological report stated that (1) the expected arrival time for the eruption was 10 years hence but (2) the the volcano could erupt at any time. These two statements make the exponential distribution a suitable probability distribution for modeling the arrival time of the eruption. An important property of the exponential distribution is that it has no memory. Knowing that the eruption did not occur during the time interval $[0,\, t]$ does not help us in predicting if the eruption will occur during the time interval $[t,\, t + \Delta t]$. In other words, the future is independent of the past. A few examples of events that are generally considered to be exponentially distributed are natural disasters, equipment failure rates, corporate bond defaults and the arrival of news that cause stock prices to jump.

We will define the random variable Z to be the arrival time of the eruption. The length of the time interval over which the pizzeria survives is therefore Z. We will define μ to be a survival parameter such that the expected value of the random arrival time Z is μ. According to the geological report...

$$\mathbb{E}\Big[Z\Big] = \mu = 10\,\text{years} \tag{2}$$

It can be shown that the expected value of the exponentially-distributed random variable Z is a function of the hazzard rate λ. The expected value of Z is...

$$\mathbb{E}\Big[Z\Big] = \frac{1}{\lambda} \tag{3}$$

A major advantage to using the exponential distribution is that there is only one variable that we need to estimate and that variable is λ, which is the hazzard rate. We can combine Equations (2) and (3) and solve for λ. The hazzard rate that we will use in our valuation is...

$$\mu = \frac{1}{\lambda}$$
$$\lambda = \frac{1}{\mu}$$
$$\lambda = 0.10 \tag{4}$$

Using an exponential distribution and the hazard rate as defined by Equation (4) above the probability that the volcano will not erupt during the time interval [0, t] is...

$$Prob\Big[No\ \ Eruption\Big] = Prob\Big[Z > t\Big] = e^{-\lambda t} \tag{5}$$

The probability that the volcano will erupt during the time interval [0, t] is...

$$Prob\Big[Eruption\Big] = Prob\Big[Z < t\Big] = 1 - Prob\Big[No\ \ Eruption\Big] = 1 - e^{-\lambda t} \tag{6}$$

The cumulative probabilities of the volcano erupting or not erupting by year are...

Table 1: Cumulative Probabilities

Year	No Eruption	Eruption
0	1.0000	0.0000
1	0.9048	0.0952
2	0.8187	0.1813
3	0.7408	0.2592
4	0.6703	0.3297
5	0.6065	0.3935
10	0.3679	0.6321
15	0.2231	0.7769
20	0.1353	0.8647
25	0.0821	0.9179
30	0.0498	0.9502

Adjusting Expected Cash Flow For Event Risk

We will make the switch to continuous time because (1) the volcano can erupt at any time and not just at year end, (2) the exponential distribution is a continuous probability distribution and (3) the owners of the pizzeria have continuous access to cash flow. To make this switch we will convert the annual cash flow growth rate and discount rate to continuous time rates. We will make the following definitions...

Table 2: Revised Valuation Assumptions

Symbol	Description	Calculation		Value
C_0	Current annualized cash flow		=	600,000
g	Continuous growth rate	$\ln(1+0.04)$	=	0.03922
k	Continuous discount rate	$\ln(1+0.20)$	=	0.18232

We will define free cash flow at any time t (F_t) to be a function of annualized free cash flow at time zero, the growth of free cash flow since time zero, the size of the time interval over which cash flow is received and whether the volcano erupted or not. We will define t to be time and δt to be an infinitesimally small change in time such that the relevant time interval from the vantage point of any time t is $[t,\, t+\delta t]$. If the volcano does not erupt prior to time $t+\delta t$ then free cash flow over the time interval $[t,\, t+\delta t]$ is...

$$F_t = C_0\, e^{gt}\, \delta t \tag{7}$$

We will define θ to be the percentage reduction in cash flows given that the eruption event has already taken place. If the volcano erupts prior to time $t+\delta t$ then free cash flow over the time interval $[t,\, t+\delta t]$ is...

$$F_t = (1-\theta)\, C_0\, e^{gt}\, \delta t \tag{8}$$

Expected free cash flow over the time interval $[t,\, t+\delta t]$ is simply the probability weighted cash flow. Given that the probabilities of eruption and no eruption are given by Equations (6) and (5), respectively, the equation for expected cash flow over the time interval $[t,\, t+\delta t]$ (uses Equations (7) and (8) above) is...

$$\begin{aligned}\mathbb{E}\Big[F_t\Big] &= C_0\, e^{gt}\, e^{-\lambda t}\, \delta t + (1-\theta)\, C_0\, e^{gt}\, (1-e^{-\lambda t})\, \delta t \\ &= C_0\, (1-\theta)\, e^{gt}\, \delta t + \theta\, C_0\, e^{gt}\, e^{-\lambda t}\, \delta t \\ &= C_0 \Big\{ (1-\theta)\, e^{gt} + \theta\, e^{(g-\lambda)t} \Big\}\, \delta t \end{aligned} \tag{9}$$

The present value of expected free cash flow over the time interval $[t,\, t+\delta t]$ (uses Equation (9) above) is...

$$\begin{aligned}\mathbb{E}\Big[F_t\, e^{-\kappa t}\Big] &= e^{-\kappa t} C_0 \Big\{ (1-\theta)\, e^{gt} + \theta\, e^{(g-\lambda)t} \Big\}\, \delta t \\ &= C_0 \Big\{ (1-\theta)\, e^{(g-\kappa)t} + \theta\, e^{(g-\kappa-\lambda)t} \Big\}\, \delta t \end{aligned} \tag{10}$$

The present value of all future cash flow (V_0) is simply the sum of the present values of expected cash flow over the time interval $[0,\, \infty]$. The equation for the present value of all future cash flow (uses Equation (10) above) is...

$$\begin{aligned} V_0 &= \int_{t=0}^{t=\infty} C_0 \Big\{ (1-\theta)\, e^{(g-\kappa)t} + \theta\, e^{(g-\kappa-\lambda)t} \Big\}\, \delta t \\ &= C_0 \Big\{ (1-\theta) \int_{t=0}^{t=\infty} e^{(g-\kappa)t}\, \delta t + \theta \int_{t=0}^{t=\infty} e^{(g-\kappa-\lambda)t}\, \delta t \Big\} \end{aligned} \tag{11}$$

The solution to the first and second integral in Equation (11) is...

$$\int_{t=0}^{t=\infty} e^{(g-\kappa)t}\, \delta t = \frac{1}{g-\kappa} e^{(g-\kappa)t} \Bigg|_{t=0}^{t=\infty} = \frac{1}{g-\kappa} \Big\{ 0-1 \Big\} = \frac{1}{\kappa-g} \tag{12}$$

$$\int_{t=0}^{t=\infty} e^{(g-\kappa-\lambda)t}\, \delta t = \frac{1}{g-\kappa-\lambda} e^{(g-\kappa-\lambda)t} \Bigg|_{t=0}^{t=\infty} = \frac{1}{g-\kappa-\lambda} \Big\{ 0-1 \Big\} = \frac{1}{\kappa-g+\lambda} \tag{13}$$

After solving the two integrals in Equation (11) the valuation equation becomes...

$$V_0 = C_0 \Big\{ \frac{(1-\theta)}{\kappa-g} + \frac{\theta}{\kappa-g+\lambda} \Big\} \tag{14}$$

Note that the cash flow growth rate g must be less then the discount rate k.

The Revised Valuation That Incorporates Event Risk

Our valuation parameters are now...

Table 3: New Valuation Parameters

Symbol	Description		Value	Reference
C_0	Annualized cash flow at year zero	=	\$600,000	Table (2)
g	Continuous cash flow growth rate	=	3.92%	Table (2)
κ	Continuous cash flow discount rate	=	18.23%	Table (2)
λ	Hazard rate	=	0.10	Equation (4)
θ	Percentage loss given eruption	=	1.00	Note: Total loss

Using Equation (14) above enterprise value that incorporates event risk is...

$$V_0 = 600,000 \left\{ \frac{(1-(1.00))}{0.1823-0.0392} + \frac{1.00}{0.1823-0.0392+0.10} \right\} = \$2,468,112 \tag{15}$$

Note that just discounting the first 10 years of cash flow overstates enterprise value by 20%. The present value of the first 10 years of cash flow is \$2,967,636.

If we assume that $\theta = 0.90$ (a 90% loss rather than a 100% loss) enterprise value that incorporates event risk is...

$$V_0 = 600,000 \left\{ \frac{(1-(0.90))}{0.1823-0.0392} + \frac{0.90}{0.1823-0.0392+0.10} \right\} = \$2,640,585 \tag{16}$$